MAT 110
Notes 3.1

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The Remainder Theorem and the Factor Theorem

Examples:

1. Divide.

$$\frac{16x^3 - 8x^2 + 12x}{4x}$$

2. Divide using long division and synthetic division.

$(6x^3 + 15x^2 - 8x + 2) \div (x + 4)$

3. Divide using synthetic division.
   $(5x^3 + 6x^2 - 8x + 1) \div (x - 5)$

4. Divide using synthetic division.
   $x^4 - 4x^2 + 7x + 15$ by $x + 4$

Remainder Theorem:
If a polynomial $P(x)$ is divided by $x-c$, then the remainder equals $P(c)$.

Examples:

1. Use the Remainder Theorem to find $P(-3)$ if $P(x) = 2x^3 - x^2 + 3x - 1$.

2. Use the Remainder Theorem to find $P(-2)$ and $P(1/2)$ if $P(x) = 2x^3 + 3x^2 + 2x - 2$.

Factor Theorem:
A polynomial P(x) has a factor (x-c) if and only if P(c) = 0. That is (x-c) is a factor of P(x) if and only if c is a zero (x-intercept) of P(x).

Examples:

1. Use synthetic division and the factor theorem to determine if the given binomial is a factor of P(x).

   $P(x) = x^3 + 4x^2 - 27x - 9$; x - 6

2. Use synthetic division and the factor theorem to determine if the given binomial is a factor of P(x).

   $P(x) = x^4 + x^3 - 21x^2 - x + 20$; x+5

Homework: pg. 290 #1, 13, 15, 17, 19, 27, 29, 37, 39, 41, 47, 49, 53, 57, 59

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Polynomial Functions of Higher Degree

Polynomial Functions | Graph

1. $P(x) = 3$

2. $P(x) = 2x - 5$

3. $P(x) = 2x^2 - 5x + 1$

$P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x^1 + a_0$ (Example: $f(x) = x^4 + x^3 + x - 1$)

What do they look like?

	n is even	**n is odd**
	$(x^2, x^4, x^6, \ldots)$	$(x^3, x^5, x^7, \ldots)$
$\mathbf{a_n > 0}$	Up to the far left and up to the far right	Down to the far left and up to the far right
$\mathbf{a_n < 0}$	Down to the far left and down to the far right	Up to the far left and down to the far right

Sketch the far-left and far-right behavior of the following functions:

1. P(x) = $-2x^3 - 6x^2 + 5x - 1$

2. P(x) = $-6x^4 - 3x^3 - 17x^2 + 2$

3. P(x) = $-x + x^4$

4. P(x) = $\frac{1}{4}(x^5 - x^4 + 2x^3 + 2x^2 - 1)$

What happens in the middle?

- Turning Points

Example:

Real Zeros (x-intercepts) of a Polynomial Function:

Factor to find the real zeros and sketch the graph.

1. $P(x) = x^3 - 6x^2 + 8x$

2. $P(x) = -x^3 - x^2 + 6x$

Even and Odd Powers of (x-c) Theorem:

If the polynomial P(x) has (x-c) as a factor k times, the graph will:

- Touch but not cross the x-axis at (c,0) if k is even and positive
- Cross the x-axis at (c,0) if k is odd and positive

Example:
Consider $P(x) = x^3 - 5x^2 - 8x + 48 = (x+3)(x-4)^2$

- x = -3 is a zero

- x = 4 is a zero

- Sketch the graph.

Example:

Determine where $P(x) = (x+3)(x-2)^2(x-4)^3$ crosses the x-axis and where it intersects (touches) the x-axis. If $P(x) = x^6 + \ldots$, sketch the graph.

Examples:

1. Sketch the graph of $P(x) = x^3 - 4x^2 + 4x$ using the theorems from section 3.2.

2. Sketch the graph of $P(x) = -x^3 - 6x^2 - 9x$ using the theorems from section 3.2.

Homework: pg. 306 #1, 3, 5, 7, 9, 11, 13, 17, 19, 33, 35, 39, 41, 43, 45

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<u>Definition of Multiple Zeros of a Polynomial Function</u>:
If a polynomial function P(x) has (x-r) as a factor exactly k times, then r is a zero of multiplicity k of the polynomial function P(x).

Examples:

1. $P(x) = x^2 + 6x + 9$

2. $P(x) = (x-5)^2(x+2)^3(x+4)$

<u>Descartes' Rule of Signs</u>:
Let P(x) be a polynomial function with real coefficients and with the terms arranged in order of decreasing powers of x.

1. The number of positive real zeros of P(x) is equal to the number of variations in sign of P(x), or to that number decreased by an even integer.
2. The number of negative real zeros of P(x) is equal to the number of variations in sign of P(-x), or to that number decreased by an even integer.

Examples:
Find the number of possible positive and negative real zeros of the polynomial functions.

1. $P(x) = 3x^3 + 11x^2 – 6x – 8$

2. $P(x) = 2x^4 – x^3 + 2x^2 + 1$

Rational Zero Theorem:

If $P(x) = a^n x_n + a^{n-1} x_{n-1} + \ldots. + a^1 x + a_0$ has **integer** coefficients and $\frac{p}{q}$ is a rational zero (in lowest terms) of P, then

- p is a factor of the constant term a_0 and
- q is a factor of the leading coefficient a_n

Examples - continued:
Find the possible rational zeroes of the polynomial functions.

1. $P(x) = 3x^3 + 11x^2 - 6x - 8$

2. $P(x) = 2x^4 - x^3 + 2x^2 + 1$

Examples:

1. Find the zeroes and sketch the polynomial function $P(x) = 2x^3 - 3x^2 - 3x + 2$.

2. Find the zeroes and sketch the polynomial $P(x) = 3x^4 - 2x^3 - 25x^2 + 28x + 12$.

Homework: pg. 321 # 1, 3, 7, 9, 11, 27, 29, 35, 41, 43, 55

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The Fundamental Theorem of Algebra:
Complex Numbers (a + bi)

- The Real Numbers are a subset of the Complex Numbers

Examples of Complex Numbers:

Examples of Real Numbers:

The number of zeros of a polynomial function:
If P(x) is a polynomial function of degree $n \geq 1$, then P(x) has exactly n complex zeros, provided each zero is counted according to multiplicity.

Example:
How many complex zeros does the function have?

$$F(x) = x^5 - x^4 + 3x^2 - 1$$

Examples:

Find the zeros and linear factors of the following polynomial functions.

1. $P(x) = x^3 - 13x^2 + 65x - 125$

2. $P(x) = x^5 + 2x^4 - 4x^3 - 12x^2 + 3x + 10$

3. $P(x) = x^5 - 9x^4 + 34x^3 - 58x^2 + 45x - 13$

Examples:
Find a polynomial that has the following zeros:

1. 2, -2, 1

2. 2i, 7

Homework: pg. 332 # 1, 3, 5, 9, 13, 37, 39, 40

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Graphs of Rational Functions and their Applications:

Consider $F(x) = \frac{x+1}{x-2}$

Find the x- and y-intercepts.

Find the domain (set the denominator equal to zero).

Sketch the graph based on this information.

Vertical Asymptote:
The line x = a is a *vertical asymptote* of the graph of the function F provided

$$F(x) \to \infty \text{ or } F(x) \to -\infty$$

as x approaches a from either the left or right.

Graphs:

Horizontal Asymptote:
The line y = b is a *horizontal asymptote* of the graph of a function F provided

$$F(x) \to b \text{ as } x \to \infty \text{ or } x \to -\infty$$

Graphs:

Theorem of Vertical Asymptotes:
If the real number a is a zero of the denominator Q(x), then the graph of F(x)=P(x)/Q(x), where P(x) and Q(x) have no common factors, has the vertical asymptote x = a.

Theorem on Horizontal Asymptotes:
Let $F(x) = \dfrac{a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0}{b_m x^m + b_{m-1}x^{m-1} + ... + b_1 x + b_0}$ be a rational function with numerator of degree n and denominator of degree m.

1. If n < m, then the x-axis, which is the line given by y=0, is the horizontal asymptote of the graph of F.

2. If n = m, then the line given by $y = a_n/b_m$ is the horizontal asymptote of the graph of F.

3. In n > m, then the graph has no horizontal asymptotes.

Example:
Find the vertical and horizontal asymptotes of each function.

a) $f(x) = \dfrac{3x^2 + 5}{x^2 - 4}$

b) $p(x) = \dfrac{3x - 5}{x^3 - 8}$

General Procedure for Graphing Rational Functions that have no Common Factors:
Let F(x) = P(x)/Q(x)

1. Asymptotes
   Find the real zeros of the denominator Q(x). For each zero a, draw the dashed line x = a. Each line is a vertical asymptote of the graph of F. Also graph any horizontal asymptotes.

2. Intercepts
   Find the real zeros of the numerator P(x). For each real zero c, plot the point (c,0). Each such point is an x-intercept of the graph of F. For each x-intercept use the even and odd powers of (x-c) to determine if the graph crosses the x-axis at the intercept of if the graph intersects but does not cross the x-axis. Also evaluate F(0). Plot (0,F(0), the y-intercept of the graph of F.

3. Complete the sketch.
   Use all the information obtained above to sketch the graph of F.

Example:

Sketch the graph of F(x) = $\frac{1}{x-2}$

Example:

Sketch the graph of F(x) = $\frac{x+3}{1-x}$

Example:

Sketch the graph of F(x) = $\dfrac{x^2}{x^2 - x - 6}$

Homework: pg. 348 #1, 3, 5, 7, 9, 15, 17, 21, 23, 25, 27, 29, 31, 33