**Lecture Notes: Approximating D.E. Solutions Using Taylor's Method**

`BACKGROUND`: Review Taylor Polynomials from Calc II before beginning this section.

It is usually hard or impossible to find an analytic solution to most ODE's. This suggests the need for methods to approximate the solutions to ODE's. For first order ODE's, Taylor's method is a very good choice.

Let $\phi_n(x)$ be the exact solution to the ODE

$$y' = f(x, y) \quad \phi_n(x_n) = y_n$$

Then the corresponding Taylor Polynomial for $\phi_n(x)$ around $x_n$ is

$$\phi_n(x) = \phi_n(x_n) + \phi_n'(x_n)(x - x_n) + \frac{\phi_n''(x_n)}{2!}(x - x_n)^2 + ...$$

Notice a smart choice for $x$ would be $x_{n+1}$. This is because now $x - x_n = x_{n+1} - x_n = h$. Also, notice that $\phi_n(x_n) = y_n$ and $\phi_n'(x_n) = f(x_n, y_n)$ from the initial conditions. So, substituting and simplifying yields...

$$\phi_n(x_{n+1}) = y_n + hf(x_n, y_n) + \frac{\phi_n''(x_n)}{2!}(h)^2 + ...$$

Now... if $\phi_n(x)$ is suppose to be substituted in for $y$ in the original ODE, then $\phi_n''$ must come from the second derivative of $y$ (or the first derivative of $y'$). Since $y'$ is a multivariate function (it equals f(x,y)), we will need to take the total differential instead of 'derivative'.

$y'' = f_x dx + f_y dy = f_x(1) + f_y(y') = f_x(x, y) + f_y(x, y)f(x, y)$

Define this $y''$ as $f_2(x, y)$. Likewise, $y''' = f_3(x, y) =$ total differential of $f_2(x, y)$, etc.

Last step... Notice $\phi_n(x_{n+1})$ is attempting to find the *next* y value at $x_{n+1}$. This implies we can approximate $\phi_n(x_{n+1})$ by $y_{n+1}$. Also, $\phi_n''$ and $\phi_n'''$ can be exchanged for $f_2(x, y)$ and $f_3(x, y)$, etc.

## Taylors Method

$x_{n+1} = x_n + h \quad (\leftarrow$ notice $x_{n+1}$ is only in terms of known previous values)

and

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!}f_2(x_n, y_n) + \frac{h^3}{3!}f_3(x_n, y_n) + ... + \frac{h^p}{p!}f_p(x_n, y_n)$$

NOTES:

· You get to pick p, and therefore how accurate you would like the approximation.

· The accuracy does not only depend on h, like Euler's Method.

· Upside: Very good accuracy / Downside: Computations get ugly fast