Chapter 9
Estimation and Confidence Intervals

This chapter is about making **inferences** about populations based on samples drawn from the populations.

If we took a number of samples from a population and calculated their mean, we would get a different mean, or point estimate, for every sample as illustrated below.

μ

Population: ________________|__________________

Sample means as point estimates

$\bar{x}$

$\bar{x}$

$\bar{x}$

$\bar{x}$

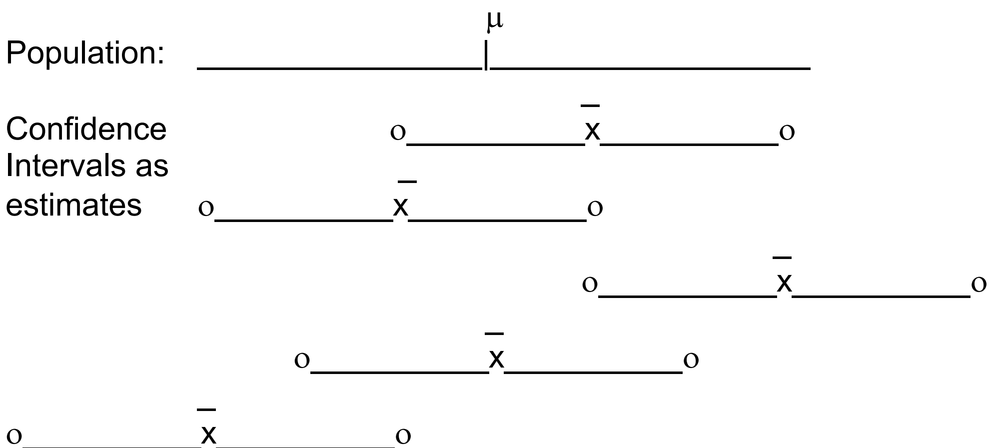
$\bar{x}$

Not only do none of the sample means equal the population mean, they're all different from each other as well. So, how can we make any inferences about the population mean from any of these sample means?

We do this by construction of a **confidence interval** about the sample mean. This interval, one number lower than $\bar{X}$ and one higher, is selected so that a high level of confidence exists that the true population mean lies between the two numbers. Hence, we give up our chance to be "exactly wrong" (the probability that any sample mean exactly equals the population mean is zero), and settle for a less exact "interval" which has a high probability of being correct.

A confidence interval is always constructed with a **level of confidence** in mind. If we want to be 95% confident that our confidence interval contains the population mean, it means that, by selecting many different samples of size n from a population, and constructing a "95% confidence interval" around the mean of each sample, then 95% of those intervals would actually contain the true population mean. Stated differently, for any sample from a population, if we construct a 95% confidence interval about the mean of the sample, that confidence interval has a **95% probability** of containing the true mean of the population from which the sample was drawn.

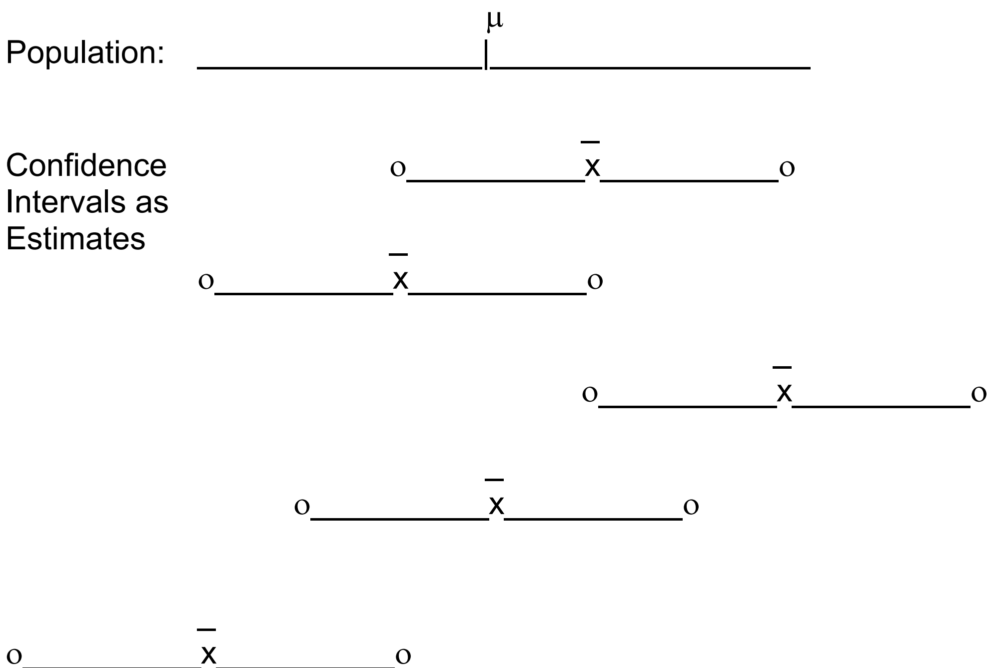
The most common values for this **degree of confidence** are 99%, 95% and 90%. Confidence intervals could be represented graphically below. Note that the intervals are centered on $\bar{X}$. We will determine the end-points of the interval by determining how many standard deviations from $\bar{X}$ they must be in order to satisfy the degree of confidence desired.



We know from the Central Limit Theorem that the distribution formed by the means of these samples will approximate a normal distribution, as drawn below.

We also know that, if we want a confidence interval at the 95% level of confidence, then the central portion of the graph covering 95% of the area (or probability) must represent this interval. The remaining 5% must therefore be located in the "tails" of the distribution. If we denote **the level of confidence as the probability that the mean is contained within the confidence interval (95%)**, and the probability that it is not contained in this interval by α, then it follows that $1 - \alpha = 0.95$. Since the distribution and the confidence interval both have symmetry about the mean, then $\alpha/2$ or .025 of the probability must reside in each "tail" of the distribution.

We can therefore say that it is very "likely" that our confidence interval contains the true population mean (95% probability) and very "unlikely" that it does not (5% probability), as shown below.



$z_{\alpha/2}$ is the z score that separates an area of $\alpha/2$ in the tail of a standard normal distribution from the rest of the distribution. $z_{\alpha/2}$ is called the critical score, and it is totally dependent on the chosen level of confidence.

Example: For a 95% confidence interval, $0.95 = 1 - \alpha$, so $\alpha = 0.05$. Then $\alpha/2$ is 0.025. Therefore, $P(0 < z < z_{\alpha/2}) = .95/2 = .475$. From the z-table, $z_{\alpha/2} = 1.96$ makes this a true statement. Thus, $z_{\alpha/2} = 1.96$, since on the standard normal distribution, the interval from $z = -1.96$ to $z = +1.96$ contains 95% of the probabilities. See graph below:

Note that ANY time that we use a confidence interval at the 95% level of confidence, the critical score will always be equivalent to $z_{\alpha/2} = 1.96$.

Note also that, any time the distribution of means in question is not the standard normal distribution (i.e., either $\mu \neq 0$ and/or $\sigma \neq 1$), it can be converted to its standard normal equivalent by using the formula

$$z_{\alpha/2} = (\bar{x} - \mu) / (\sigma / \sqrt{n})$$

where n = sample size, μ = the population mean, σ = the population standard deviation, and $\bar{X}$ = the sample mean.

Examples: For $\alpha = 0.01$, find $z_{\alpha/2}$ and draw the confidence interval.

For $\alpha = 0.02$, find $z_{\alpha/2}$ and draw the confidence interval.

For $\alpha = 0.05$, find $z_{\alpha/2}$ and draw the confidence interval.

For $\alpha = 0.10$, find $z_{\alpha/2}$ and draw the confidence interval.

These results can be summarized in the table below:

Conf. Level	$z_{\alpha/2}$
.99	2.575
.98	2.33
.95	1.96
.90	1.645

Note that, if we want to be more confident that our interval includes the true population mean, our interval must be bigger (include a larger number of standard deviations). That is $z_{\alpha/2}$ must be greater. If we want to be 100% confident, than $z_{\alpha/2}$ must be infinite (very large.)

Margin of Error

We can never determine the true population mean μ in this way. However, we can determine how far X is from μ: $(\bar{X} - \mu)$. That is, we can determine how wide the interval is. The width of this interval, therefore, is called the **margin of error** and is denoted by the letter **E.** Hence, **E = $(\bar{X} - \mu)$**. **[Note: Your text author does not use the terminology "margin or error" at this point of the text. However, it is the exact concept that he is using, as we will see later. The text calls E the "allowable error"].**

By writing the equation $z_{\alpha/2} = (\bar{x} - \mu) / (\sigma/\sqrt{n})$, to find

$$(\bar{x} - \mu) = z_{\alpha/2} (\sigma/\sqrt{n})$$

then the margin of error E is not only $\bar{X} - \mu$, it is also

$$E = z_{\alpha/2} (\sigma/\sqrt{n})$$

That is, E is the number of standard deviations from the mean ($z_{\alpha/2}$, determined by the desired level of confidence), times the standard deviation of the distribution of sample means $(\sigma/\sqrt{n})$.

Therefore, there is a probability of α that the mean $\bar{x}$ of any sample will be in error (different from μ) by more than E. This leads to the result that the confidence interval for the population mean μ is given by the formula

$$(\bar{x} - E) < \mu < (\bar{x} + E)$$

with $\bar{x} - E$ and $\bar{x} + E$ being the **confidence interval limits** determined for a particular level of confidence.

Since we rarely know the true standard deviation σ of a population, we can **replace σ by the sample standard deviation "s" if n $\geq$ 30.** If n < 30, then the population must have a normal distribution and we must know σ to use the above formula.

If we continue to draw samples from a population and construct confidence intervals from their sample means based on a selected α, it is correct to say that over the long run (1-α)% of the intervals will contain μ and α% won't.

This is illustrated by the graph below:

Example: For the 95% confidence level, find $z_{\alpha/2}$ and the confidence interval for μ if $\sigma = 5$, $x = 100$, and $n = 40$. What is the difference between the margin of error E and the confidence interval?

Homework: p. 253, Exercises 1, 3, 5, 7

t-Distributions

If we intend to construct confidence intervals but do not know σ and have a small sample size $n < 30$, $z_{\alpha/2}$ from the z-table will give inaccurate results. Another table, called the **Student's t-distribution**, has been developed to get around this problem.

If ALL of the following conditions are met

1) The distribution is essentially normal
2) $n < 30$
3) σ is unknown (must estimate using s from the sample)

then we can find the confidence interval E using the formula

$$\mathbf{E = t_{\alpha/2} s/\sqrt{n}}$$

Where $t_{\alpha/2}$ is determined from Appendix F, **s** is the sample standard deviation, and **n** is the sample size **< 30**.

The value of $t_{\alpha/2}$ is determined by entering the Table in the column designated by the appropriate α and going to the row represented by the **degrees of freedom (df).** The degrees of freedom is defined as **n-1.**

The t-distribution has the following significant properties:

1. The distribution is different for different sample sizes.
2. The distribution is generally symmetrical, but has greater variability due to smaller sample size.
3. The mean is t = 0
4. The standard deviation varies but is greater than 1.
5. As the sample size gets larger, the t-distribution more nearly resembles the standard normal distribution. When $n \geq 30$, the differences are so small that we can just use the z-score.
6. The t-distribution is unique for each α and degree of freedom. Hence, unlike the z distribution, a series of t-distributions are required.
7. The t-score is used exactly like the z-score. In formulas, "z" is simply replaced by "t" and the value for t is used. Also, note that to use the t-distribution, the value of σ cannot be known. Therefore, **s**, the sample standard deviation, is substituted for σ.

Examples: For $\alpha = 0.01$, n = 15, find $t_{\alpha/2}$ and draw the confidence interval.

For $\alpha = 0.02$, n = 15, find $t_{\alpha/2}$ and draw the confidence interval.

For $\alpha = 0.05$, n = 15, find $t_{\alpha/2}$ and draw the confidence interval.

For $\alpha = 0.10$, n = 15, find $t_{\alpha/2}$ and draw the confidence interval.

<u>Example:</u> For the 95% confidence level, find $t_{\alpha/2}$ and the confidence interval for μ if s = 5, $\bar{x}$ = 100, and n = 18.

Homework: p. 260, Exercises 9, 11, 13

Proportions

We can apply the concept of margin of error to **binomial distributions**, as they represent proportions or percentages of populations (π = percentage of population that is success, and (1- π) = percentage of population that is failure).

Remember, a binomial distribution has a fixed number of trials, only two outcomes (success and failure) and the probability of success or failure is known and constant. Also, **if $n\pi \geq 5$ and $n(1-\pi) \geq 5$, then it is possible to use a normal distribution as an approximation of the binomial distribution and the following discussion will apply.**

If we only have one trial, then x becomes the number of successes in that trial and x/n becomes the proportion (%) of successes in our trail of size n. We call that proportion “p”. In other words, p is the percentage of successes “x” in a sample of size “n”. It represents the **point estimate of the true population percentage π,** much like the sample mean is a point estimate of the true population mean.

The following formula follows:

p = x/n (note: p is never greater than 1)

where p is the sample percentage of x success out of n items. It also follows that the percentage of non-successes is 1 - p.

Example: Suppose a survey of 2,000 adult Americans shows that 900 favor the death penalty. Then the proportion of adults in the sample that favor of the death penalty is

$$p = 900/2{,}000 = 0.45$$

and we consider p to be the best point estimate of the true **population proportion π** that support the death penalty.

Remember that for a binomial distribution of n trials, the population mean and standard deviation are found by

$$\mu = n\pi \text{ and } \sigma = \sqrt{n\pi(1-\pi)}$$

From the single sample, we don't know the exact π and 1- π, but we do know the point estimates, which are p and and 1-p. Therefore, we can write the margin of error formula as

$$E = z_{\alpha/2}\sqrt{p(1-p)/n}$$

And the confidence interval is

$$p - E < \pi < p + E$$

A survey of 2,000 adults shows 1,280 have money in a regular savings account. Find the 95% confidence interval for the true proportion of adults who have money in a regular savings account.

x =
n =

p = x/n =
1 – p =
$z_{\alpha/2}$ =

Then $E = z_{\alpha/2}\sqrt{p(1-p)/n}$

and the confidence interval is:

Or, the proportion of the population with money in regular is savings accounts is ____________ with a margin of error of _____________ at the 95% confidence level.

Note that the margin of error E for proportions is always between 0 and 1, since proportions are stated as percentages of the whole.

Homework: p. 263, Exercises 15, 17

Sample Size

From the equation for margin of error E, it is possible to see that the margin of error in an experiment depends only on the population standard deviation σ, the confidence level we desire, α, and the sample size, n. Therefore, if we determine the margin of error **E** we desire, the sample size **n** required to achieve that **E** can be calculated. This will give us the results we require without the need to incur large expenses associated with large samples or to incur unacceptable error if the sample is too small.

Manipulating the equation for E using algebra, we see that the sample size needed for any level of confidence and margin of error can be determined from the equation

$$\mathbf{n = ((z_{\alpha/2}\sigma)/E)^2}$$

Note that the size of sample needed does NOT depend on the size of the population. Hence, it is possible to make inferences about any size population from polls with at most a few hundred respondents.

Example: For the 95% confidence level, find $z_{\alpha/2}$ and the sample size necessary to assure a maximum allowable error, E, to be no more than +/- 0.5 for an estimate of μ in a population with $\sigma = 5$. What sample size would be required if we wanted an allowable error of no more than +/- 0.2?

Note that any “n” so calculated must **always be rounded up** to the next highest integer. If it is rounded down, the “n” will not be sufficient to ensure the required margin of error E.

Following the same concept, we can solve the formula for E if the data is in for form of proportions. When we do so, the formula for sample size "n" given a required margin of error "E" follows:

$$\mathbf{n = p(1-p)(z_{\alpha/2}/E)^2}$$

Unknown σ and s

If σ and s are unknown, as is sometimes the case, an estimate can be used by determining the σ using the range rule of thumb, or $\sigma \cong$ range/4. Though this will diminish the accuracy of the results, it may still allow probabilities to be calculated.

Homework: p. 267, Exercises 23, 25, 27, 29
pp. 269-272, Exercises 31, 33, 35, 41, 45, 49, 53, 55, 57